Filtrations and associated graded rings and modules

In this section, we give some commutative algebraic definitions that will help us construct some important geometric objects such as the blowup algebra and the tangent cone.

These constructions involve a <u>multiplicative filtration</u> of a ring R. That is, a sequence of ideals

$$R = I_0 \supset I_1 \supset \dots \quad \text{s.t.} \quad I_i I_j \subset I_{i+j} \text{ for all } i, j.$$

Usually we'll care about the case where $I \subseteq R$ is an ideal and $I_i = I^i$, called the <u>I-adic filtration</u>.

We can also generalize this: $M \supset IM \supset I^2M \supset \dots$ is the I-adic filtration of the R-module M. Even more generally:

Def: A filtration $M = M_0 \supset M_1 \supset \dots$ is called an <u>I-filtration</u> if $IM_n \subset M_{n+1}$ $\forall$ $n \geq 0$.

An I-filtration is <u>I-stable</u> if in addition $IM_n = M_{n+1}$ for $n >> 0$.

We will later prove the Artin-Rees Lemma, which says that an I-stable filtration of a finitely generated R-module is still I-stable when intersecting w/ a f.g. submodule.

Associated graded rings and modules

Let $I \subseteq R$ be an ideal. The <u>associated graded ring of R w.r.t. I</u>, is

$$gr_I R := R/I \oplus I/I^2 \oplus \dots$$

The multiplication is given as follows: if $\bar{a} \in I^m/I^{m+1}$ and $\bar{b} \in I^n/I^{n+1}$, s.t. $a \in I^m$, $b \in I^n$, define $\bar{a}\bar{b} \in I^{m+n}/I^{m+n+1}$ to be the image of ab.

We need to verify that this is well-defined: If $a' \in I^m$ and $b' \in I^n$ have images $\bar{a}$ and $\bar{b}$ in I^m/I^{m+1} and I^n/I^{n+1} respectively, then $a' = a + x$ and $b' = b + y$, $x \in I^{m+1}$, $y \in I^{n+1}$

$$\Rightarrow a'b' = ab + \underbrace{ay + bx + xy}_{\in\, I^{m+n+1}}$$

so $a'b'$ and ab are equivalent mod I^{m+n+1}.

More generally if $\mathcal{Y}: M = M_0 \supset M_1 \supset \dots$ is an I-filtration of an R-module M, define

$$gr_{\mathcal{Y}} M := M/M_1 \oplus M_1/M_2 \oplus \dots$$

This is a $gr_I R$ module as follows: If $\bar{a} \in I^m/I^{m+1}$, $\bar{b} \in M_n/M_{n+1}$, w/ $a \in I^m$, $b \in M_n$ lifts, then $ab \in I^m M_n \subseteq M_{n+m}$

so $\bar{a}\bar{b} \in M_{n+m}/M_{n+m+1}$ is the image of ab.

We now see why stability of a filtration is important:

Prop: Let $I \subseteq R$ be an ideal, and M a finitely generated R-module. If

$$\mathcal{J}: M = M_0 \supset M_1 \supset \dots$$

is an I-stable filtration w/ M_i f.g. $\forall i$, then $gr_{\mathcal{J}} M$ is a finitely generated $gr_I R$-module.

Pf: Assume $IM_i = M_{i+1}$ for $i \geq n$. Then

$$\left(I/I^2\right)\left(M_i/M_{i+1}\right) \subseteq M_{i+1}/M_{i+2} = IM_i/I^2M_i .$$

But if $\overline{rm} \in IM_i/I^2M_i$, w/ $r \in I$, $m \in M_i$, then $\overline{r} \cdot \overline{m} \in \left(I/I^2\right)\left(M_i/M_{i+1}\right)$.

so they are equal.

Thus, the unions of the generators of $M_0/M_1, \dots, M_n/M_{n+1}$ generate $gr M$.

But each M_i is f.g., so $gr M$ is as well. $\square$

We don't really have any interesting maps $M \longrightarrow gr M$, but we do have a natural set map:

Let $\mathcal{J}$ be the filtration $M = M_0 \supset M_1 \supset \dots$, and take $f \in M$.

If there is some m s.t. $f \in M_m$ but $f \notin M_{m+1}$, define the <u>initial form of f</u> to be

$$in(f) := \overline{f} \in M_m/M_{m+1} \subset gr M.$$

Otherwise, if $f \in \bigcap_0^\infty M_m$, then define $\operatorname{in}(f) = 0$.

Ex: Let $J = (xy + y^3, x^2) \subseteq R = k[x,y]$, and $I = (x,y)$, and consider the I-adic filtration of R.

Define $\operatorname{in}(J)$ to be the $\operatorname{gr}_I R$-submodule of R generated by $\operatorname{in}(f)$ for all $f \in J$.

Note that $\operatorname{in}(x^2) = x^2 \in I^2/I^3$, and $\operatorname{in}(xy + y^3) = xy \in I^2/I^3$.

However $x(xy + y^3) - yx^2 = xy^3 \in J$, so $y^2(xy+y^3) - xy^3 = y^5 \in J$, so $y^5 \in \operatorname{in}(J)$ but y^5 is not generated by x^2 and xy in $\operatorname{gr}_I R$.

So you can't find $\operatorname{in}(J)$ by looking at the images of its generators.

The nice thing about this construction is that it gives us a way to turn an arbitrary Noetherian rings into finitely generated graded algebras:

Let $I \subseteq R$ be a max'l ideal, R Noetherian. Then

$$\operatorname{gr}_I R = \underbrace{R/I}_{= k} \oplus I/I^2 \oplus \cdots$$

and $I = (f_1, \dots, f_n)$, so for $a \in I/I^2$, $a = r_1 f_1 + \dots + r_n f_n$ where $r_i = 0$ or $r_i \notin I$.

If $a \in I^m/I^{m+1}$, $a = r_1 f_1 + \dots + r_n f_n$, where each $r_i \in R \setminus I^m$ or $r_i = 0$ so by induction r_i is a polynomial in the f_i over k.

This gives us a well-defined Hilbert function for local Noetherian rings:

Def: If R is a local ring w/ maximal ideal I, then the <u>Hilbert function</u> of R is

$$H_R(n) = \dim_{R/I} I^n/I^{n+1}.$$

If M is a f.g. R-module, define

$$H_M(n) = \dim_{R/I} I^n M / I^{n+1} M.$$

Note that these are just the Hilbert functions of $gr_I R$ and $gr_I M$, so we already know that for large values of n the agree w/ polynomials $P_R(n)$ and $P_M(n)$ of deg $\leq H_R(1) - 1$.

A lot of the time, we can learn about R by looking at $gr_I R$. However, we need to make sure no elements of R are lost in $gr_I R$. i.e., we need that $\bigcap_j I^j = 0$.

We will soon see (by the Krull intersection theorem) that this is usually the case.